Anthias Liquip Score: Measuring Liquidation Probability

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1. Introduction

Decentralized Finance (DeFi) lending protocols use a “health factor” metric to decide if a wallet can be liquidated. This metric is defined as the ratio of risk-weighted collateral value to the risk-weighted loan value. When the health factor drops below a set threshold, typically 1, liquidation becomes a possibility. This can be expressed mathematically as:

$$H_f = \frac{\sum_{i=1}^{n}(c_i \times c_{f_i})}{\sum_{i=1}^{m}(d_i / d_{f_i})}$$

In this formula, c_i and d_i are the collateral and debt amounts of asset i in a standard unit like Ethereum. The “collateral factor” and “debt factor” (c_{f_i} and d_{f_i}) are risk weights given to the collateral and debt, and typically depend on the asset’s volatility.

While useful, the health factor doesn’t provide a full picture of risk. For example, a self-collateralized position, where the loan’s health factor is close to 1 and the debt and collateral are in the same asset, has a liquidation risk tied only to the interest accrued on the debt. In contrast, a wallet with a much higher health factor can still face a significant liquidation risk due to the volatility of two different assets and the accruing interest on the debt.

Given that the health factor may not fully capture these complexities, it can lead to misleading risk assessments. This paper introduces Anthias’ method for calculating a different metric called the “Liquip Score” (liquidation probability score). By considering the variances and correlations of the price changes of assets in a lending position, this score provides a more accurate measure of liquidation risk.

Additionally, we introduce the “Days Until Liquidation” concept. This provides an estimate of the time left until a position might face liquidation under current market conditions, adding another layer of understanding to our risk assessments. These improved methods allow for a clearer view of liquidation health for individual positions and the protocol as a whole.

2. Obtaining Returns Distribution

The returns of a wallet position can be conceptualized as a random variable. The distribution of these returns can be characterized by its mean and variance, which collectively provide a comprehensive understanding of the wallet position’s expected performance and associated risk. The following sections detail the calculation of the mean and variance to elucidate the returns distribution for a wallet position.

2.1 Calculating Covariance Matrix

Assuming a wallet supplies n assets and borrows m assets, the price of over $k+1$ days of each collateral and debt asset can be represented as vectors with $k+1$ entries $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, ..., \mathbf{c}^{(n)}$ and $\mathbf{d}^{(1)}, \mathbf{d}^{(2)}, ..., \mathbf{d}^{(m)}$, respectively. Correspondingly, the returns of each collateral and debt asset over k days are represented as vectors with k entries $\mathbf{c}_r^{(1)}, \mathbf{c}_r^{(2)}, ..., \mathbf{c}_r^{(n)}$ and $\mathbf{d}_r^{(1)}, \mathbf{d}_r^{(2)}, ..., \mathbf{d}_r^{(m)}$, respectively, where each entry in $\mathbf{c}_r^{(i)}$ and $\mathbf{d}_r^{(l)}$ is defined as:

$$c^{(i)}_{r_{j-1}} = ln(\frac{c^{(i)}_j}{c^{(i)}_{j-1}}), \quad j = 2, 3, ..., k+1$$

$$d^{(l)}_{r_{j-1}} = -ln(\frac{d^{(l)}_j}{d^{(l)}_{j-1}}), \quad j = 2, 3, ..., k+1$$

Note that the entries of $\mathbf{d}^{(l)}_r$ have negative returns when compared to $\mathbf{c}^{(i)}_r$ because $\mathbf{d}^{(l)}_r$ represents the returns of a debt position.

The mean returns of each asset can then be expressed as scalars $\mu_{\mathbf{c}^{(1)}_r}, \mu_{\mathbf{c}^{(2)}_r}, ..., \mu_{\mathbf{c}^{(n)}_r}$ and $\mu_{\mathbf{d}^{(1)}_r}, \mu_{\mathbf{d}^{(2)}_r}, ..., \mu_{\mathbf{d}^{(m)}_r}$, respectively, where $\mu_{\mathbf{x}^{(i)}_r}$ is defined as:

$$\mu_{\mathbf{x}^{(i)}_r} = \frac{1}{k} \sum_{j=1}^{k} x_{r_j}$$

Subsequently, the mean deviation returns of each asset are represented as vectors with k entries $\boldsymbol{\delta}_{\mathbf{c}^{(1)}_r}, \boldsymbol{\delta}_{\mathbf{c}^{(2)}_r}, ..., \boldsymbol{\delta}_{\mathbf{c}^{(n)}_r}$ and $\boldsymbol{\delta}_{\mathbf{d}^{(1)}_r}, \boldsymbol{\delta}_{\mathbf{d}^{(2)}_r}, ..., \boldsymbol{\delta}_{\mathbf{d}^{(m)}_r}$, respectively, where $\boldsymbol{\delta}_{\mathbf{x}^{(i)}_r}$ is defined as:

$$\boldsymbol{\delta}_{\mathbf{x}^{(i)}_r} = \mathbf{x}^{(i)}_r - \mu_{\mathbf{x}^{(i)}_r}$$

Now, define $\boldsymbol{\Delta}$ as a $k \times (n+m)$ matrix with the mean deviation returns of each asset as its columns:

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\delta}_{\mathbf{c}^{(1)}_r} & \boldsymbol{\delta}_{\mathbf{c}^{(2)}_r} & \cdots & \boldsymbol{\delta}_{\mathbf{c}^{(n)}_r} & \boldsymbol{\delta}_{\mathbf{d}^{(1)}_r} & \boldsymbol{\delta}_{\mathbf{d}^{(2)}_r} & \cdots & \boldsymbol{\delta}_{\mathbf{d}^{(m)}_r} \end{bmatrix}$$

Finally, define $\boldsymbol{\Omega}$ as a $(n+m) \times (n+m)$ matrix where the (i, j) entry is the covariance between the i^{th} and j^{th} assets:

$$\boldsymbol{\Omega} = \frac{1}{k-1} \boldsymbol{\Delta}^T \boldsymbol{\Delta}$$

2.2 Calculating Wallet Variance

A position vector $\mathbf{v}$ is defined as a vector with $n+m$ entries where entry v_i is the proportion of the total value of the position that is comprised of asset i. Let us define the total value of the position, § as:

$$\S = \S_c + \S_d$$

where $\S_c$ and $\S_d$ are the total risk weighted values of the collateral and debt, respectively. Thus, $\S_c$ and $\S_d$ can be expressed as:

$$\S_c = \sum_{i=1}^{n} c_i \times c_{f_i}, \quad \S_d = \sum_{j=1}^{m} d_j / d_{f_j}$$

where c_i and c_{f_i} are the collateral amount and collateral factor of asset i, respectively, and d_j and d_{f_j} are the debt amount and debt factor of asset i, respectively. These values are more clearly defined in the introduction.

Thus, each entry of the position vector $\mathbf{v}$ may be expressed as:

$$v_i = \frac{c_i \times c_{f_i} + d_i / d_{f_i}}{\S}$$

Keep in mind that any asset i will be either debt or collateral, so either c_i or d_i will be 0. Thus, the position vector $\mathbf{v}$ is a vector of proportions that sum to 1.

Finally, the wallet variance σ^2 is a scalar defined as:

$$\sigma^2 = \mathbf{v}^T \mathbf{\Omega} \mathbf{v}$$

σ^2 represents the variance of the combined returns for the wallet position on a single day, considering the proportions of assets, their historical returns, and their correlations. This metric quantifies the overall risk associated with a wallet position in a DeFi lending protocol, offering a valuable risk assessment tool that provides a quantitative measure of the overall volatility of the combined returns for the wallet position while incorporating the correlation structure of the assets involved. By accounting for the variance and correlations between the assets in a position, wallet variance captures a more comprehensive assessment of the risk involved, leading to a more accurate representation of risk exposure. As a result, this metric can be particularly useful for determining a wallet's risk in relation to liquidation, as will be discussed under "Liquip" below.

2.3 Calculating Mean Returns

In order to fully describe the distribution of returns for a wallet position, the mean returns of the position must be calculated.

First, the mean returns for each asset $\boldsymbol{\mu}_r$ is defined as a vector with $n + m$ entries defined as:

$$\boldsymbol{\mu}_r^T = \begin{bmatrix} \mu_{\mathbf{c}_r^{(1)}} & \mu_{\mathbf{c}_r^{(2)}} & \cdots & \mu_{\mathbf{c}_r^{(n)}} & \mu_{\mathbf{d}_r^{(1)}} & \mu_{\mathbf{d}_r^{(2)}} & \cdots & \mu_{\mathbf{d}_r^{(m)}} \end{bmatrix}$$

Second, we must define a vector for the daily interest rates of each asset $\mathbf{r}$, which is a vector with $n + m$ entries defined as:

$$\mathbf{r}^T = \begin{bmatrix} r_{\mathbf{c}_r^{(1)}} & r_{\mathbf{c}_r^{(2)}} & \cdots & r_{\mathbf{c}_r^{(n)}} & r_{\mathbf{d}_r^{(1)}} & r_{\mathbf{d}_r^{(2)}} & \cdots & r_{\mathbf{d}_r^{(m)}} \end{bmatrix}$$

where $r_{\mathbf{c}_r^{(i)}}$ and $r_{\mathbf{d}_r^{(j)}}$ are the daily interest rates of collateral asset i and debt asset j, respectively. Any collateral asset i will have a positive daily interest rate which reflects the returns generated from supplying tokens, and any debt asset j will have a negative daily interest rate which reflects the debt accruement from borrowing tokens.

To obtain the mean returns of the position, the dot products of the position vector $\mathbf{v}$ with the mean returns of each asset $\boldsymbol{\mu}_r$, and the daily interest rates of each assets $\mathbf{r}$ are summed:

$$\mu = \mathbf{v}^T \boldsymbol{\mu}_r + \mathbf{v}^T \mathbf{r}$$

This operation yields a scalar μ that represents the mean of the combined returns for the wallet position on a single day, considering the proportions of assets and their historical returns, as well as the daily interest rates of the debt assets.

3. Usage

Building on the foundational parameters of wallet variance (σ^2) and mean returns (μ), our approach enables the calculation of the Anthias Liquip Score. This score provides a probabilistic measure of the likelihood that the value of a wallet's position falls below a certain threshold within t days. Rooted in the assumption of log-normal distribution for the value of the position, our methodology presumes that daily returns on the position adhere to a normal distribution. This underpinning allows us to leverage the framework of geometric Brownian motion, a standard tool in financial modelling, to compute the probability of a position's value dropping below a predetermined threshold within a specified timeframe.

Our model employs the assumption of log-normal distribution for asset prices, leading to an ensuing normal distribution of returns. This assumption is not without its considerations. It does, however, provide a simple and streamlined lens through which asset price behavior can be examined, simplifying the calculation of risk metrics. The normal distribution, being mathematically tractable and widely understood, offers versatility for numerous financial analyses.

However, we recognize the inherent limitations in this approach. Empirical studies have indicated that asset returns frequently exhibit characteristics such as skewness and kurtosis, thereby deviating from a perfect normal distribution. Specifically, asset returns often display 'fat tails,' indicating a greater probability of extreme events than what a normal distribution would predict. In addition, asset returns may also show signs of time-varying volatility and autocorrelation, both of which are not accounted for in the normal distribution assumption.

Despite these potential caveats, our proposed Anthias Liquip Score, rooted in the assumption of normally distributed returns, still provides valuable risk insights. It remains a practical tool, offering meaningful risk assessment as long as users remain aware of its assumptions and carefully interpret its results in light of these known trade-offs. Furthermore, our model's ability to predict 'days until liquidation' adds an additional layer of utility, equipping users with an estimate of the time horizon for potential liquidation events.

3.1 Liquip Score

The objective is to determine the Liquip Score, which is probability of a wallet's position declining by a certain amount, which would trigger liquidation, in t days. This specific amount, referred to as the "liquidation buffer," is denoted by ϕ and is defined as follows:

$$\phi = \S_c - \S_d$$

where $\S_c$ and $\S_d$ are the total risk weighted values of the collateral and debt, respectively. ϕ is the amount by which the value of the collateral must decline or the debt must increase in order to trigger liquidation. Therefore, if $S(t)$ denotes the value of the wallet's position at time t, then if $S(t) < \S - \phi$, the wallet is eligible for liquidation.

We may model this value at time t using geometric Brownian motion as follows:

$$S(t) = \S \times e^{(\mu - \sigma^2/2)t + \sigma W(t)}$$

where $S(t)$ is the wallet position value at time t, $\S$ is the initial position value, μ and σ^2 are the expected and variance of returns for the wallet position, and $W(t)$ is a Wiener process. The Wiener process is a continuous-time stochastic process that is normally distributed with a mean of 0 and a variance of t. In this context, the Wiener process is used to model the random fluctuations in the value of the wallet position over time. Also keep in mind that because our variance and expected returns are calculated on a daily basis, t is measured in days.

We may now find the probability that the wallet's position value, $S(t)$, is below $\S - \phi$ in t days by computing the following:

$$Pr(S(t) < \S - \phi) = Pr(\S \times e^{(\mu - \sigma^2/2)t + \sigma W(t)} < \S - \phi) = Pr(e^{(\mu - \sigma^2/2)t + \sigma W(t)} < \frac{\S - \phi}{\S})$$

$$= Pr((\mu - \frac{\sigma^2}{2})t + \sigma W(t) < \ln(\frac{\S - \phi}{\S})) = Pr(W(t) < \frac{\ln(\frac{\S - \phi}{\S}) - (\mu - \frac{\sigma^2}{2})t}{\sigma})$$

Keep in mind that $W(t)$ is normally distributed with a mean of 0 and a variance of t. We may therefore standardize the above expression by subtracting the mean and dividing by the standard deviation. Let Z denote the standardized random variable:

$$= Pr(Z < \frac{\ln(\frac{\S - \phi}{\S}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}})$$

Let $\Phi(z)$ denote the cumulative distribution function of the standard normal distribution. We may now express the probability of liquidation within t days as follows:

$$Pr(\text{liquidation}) = \Phi(\frac{\ln(\frac{\S - \phi}{\S}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}})$$

This probability of liquidation is the Liquip Score. Keep in mind that the Liquip Score takes in two parameters: "days backward", and "days forward". The "days backward" parameter is the number of days to use when calculating the variance and expected returns of the wallet position, given by k in this paper. The "days forward" parameter is the number of days to use when calculating the probability of liquidation, given by t. The Liquip Score is calculated by first calculating the variance and expected returns of the wallet position using the "days backward" parameter, and then calculating the probability of liquidation using the "days forward" parameter.

3.2 Days Until Liquidation

While the Liquip Score is a valuable measure of potential liquidation risk, a more pragmatic and actionable metric is often desired: the number of days until liquidation. This measure can offer a more concrete understanding of the timeframe for potential liquidation, enabling wallet holders and risk managers to make more informed decisions regarding risk mitigation strategies.

The "Days Until Liquidation" metric solves for the time t in the Liquip Score equation that results in a given probability of liquidation, denoted α. Therefore, this metric reflects the number of days until the wallet's Liquip Score reaches the specified α. This can be seen as solving the inverse problem of the Liquip Score. Instead of providing a probability of liquidation for a given timeframe, it gives the timeframe until a specified probability of liquidation is reached. It's important to note that, just like the Liquip Score, the "Days Until Liquidation" depends on the "days backward" parameter, which is used to calculate the variance and expected returns of the wallet position.

In this section, we'll approach this problem by first presenting an analytic solution and subsequently proposing a numerical method. These methods should be seen as complementary, each with their own advantages and limitations. The analytic solution can offer explicit formulas, but may sometimes yield complex or negative solutions that do not align with real-world interpretations. On the other hand, the numeric solution, although less precise, is more robust, always returning real-valued results, and can handle a broader range of cases.

3.2.1 Analytic Solution

Let α denote the probability of liquidation, and let t denote the number of days until liquidation. α is the Liquip Score, and in this instance serves as a constant which reflects a fixed probability of liquidation. For example, if $\alpha = 0.05$, then we are solving for the number of days until the wallet's Liquip Score is 5%.

We may now solve for t as follows:

$$\alpha = \Phi(\frac{\ln(\frac{\S-\phi}{\S}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}) \Rightarrow \Phi^{-1}(\alpha) = \frac{\ln(\frac{\S-\phi}{\S}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$\Rightarrow \Phi^{-1}(\alpha)\sigma\sqrt{t} = \ln(\frac{\S-\phi}{\S}) - (\mu - \frac{\sigma^2}{2})t$$

Squaring both sides yields:

$$\Rightarrow (\Phi^{-1}(\alpha)\sigma)^2 t = (\ln(\frac{\S-\phi}{\S}))^2 - 2(\mu - \frac{\sigma^2}{2})t\ln(\frac{\S-\phi}{\S}) + (\mu - \frac{\sigma^2}{2})^2 t^2$$

$$\Rightarrow 0 = (\mu - \frac{\sigma^2}{2})^2 t^2 - 2(\mu - \frac{\sigma^2}{2})t\ln(\frac{\S-\phi}{\S}) + (\ln(\frac{\S-\phi}{\S}))^2 - (\Phi^{-1}(\alpha)\sigma)^2 t$$

$$\Rightarrow 0 = (\mu - \frac{\sigma^2}{2})^2 t^2 + (-2(\mu - \frac{\sigma^2}{2})\ln(\frac{\S-\phi}{\S}) - (\Phi^{-1}(\alpha)\sigma)^2)t + (\ln(\frac{\S-\phi}{\S}))^2$$

Let $a = (\mu - \frac{\sigma^2}{2})^2$, $b = -2(\mu - \frac{\sigma^2}{2})\ln(\frac{\S-\phi}{\S}) - (\Phi^{-1}(\alpha)\sigma)^2$, and $c = (\ln(\frac{\S-\phi}{\S}))^2$. We now have a quadratic equation for t:

$$0 = at^2 + bt + c \Rightarrow t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This solution for t provides the analytic solution for the number of days until a wallet reaches Liquip Score α. Because the solution is quadratic, there are two solutions for t. We can not guarantee that these solutions are not negative or complex numbers, in which case they do not coincide to a number of days.

Ideally, this quadratic equation will yield a real positive and a real negative solution for t, in which case we may take the real positive solution as the number of days until liquidation. If the quadratic equation yields two real positive solutions, then we have two possible numbers of days until liquidation, and we may take the smaller of the two.

Cases where the quadratic equation yields no real valued solution may arise in “self collateralized positions”, where the debt and collateral are made up of the same assets. In this case, the wallet will never be eligible for liquidation by account of the price variance of the asset and will therefore never reach a Liquip Score of α. In this case, we may set the number of days until liquidation to ∞.

Cases with two negative solutions may arise in “self collateralized positions” as well, but may also arise in cases where the wallet is already in liquidation. In this case, the wallet will never be eligible for liquidation by account of the price variance of the asset and will therefore never reach a Liquip Score of α. In this case, we may set the number of days until liquidation to 0.

The analytic solution provides an explicit formula for the number of days until a wallet reaches a given Liquip Score α. This method is precise and quick, but it has its limitations. Specifically, it can return complex or negative results which do not align with our physical interpretation of time. Moreover, edge cases can arise due to the nature of the portfolios, such as “self collateralized positions” where the debt and collateral are made up of the same assets. Despite these limitations, the analytic solution can provide a good starting point and an intuitive understanding of the factors affecting the time to liquidation.

3.2.2 Numeric Solution

The analytic solution has several edge cases that may be difficult to handle. In order to avoid these edge cases, we may use a numeric solution to calculate the number of days until liquidation. The numeric solution is given by using a root finding algorithm to find the value of t that satisfies the following equation:

$$f(t) = \Phi(\frac{\ln(\frac{\S-\phi}{\S}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}) - \alpha = 0$$

In order to find the value of t that satisfies the equation, we can employ the bisection method, which is a root-finding algorithm that determines the root of a function within a specified interval. The algorithm begins by examining the interval $[0, t_{\text{max}}]$, and checks whether a root is guaranteed in this interval by examining the signs of the function at the interval endpoints. If $f(0) \times f(t_{\text{max}}) > 0$, the function does not change sign over the interval and there is no guaranteed root in the interval $[0, t_{\text{max}}]$.

If a root is guaranteed, the algorithm continues by dividing the interval into two by finding the midpoint. It then determines which subinterval guarantees a root and continues to narrow down the interval by choosing the subinterval that contains the root, and then again finding the midpoint of this subinterval. This process is repeated until the function value at the midpoint is sufficiently close to zero (a chosen small tolerance ϵ) or the width of the interval is sufficiently small (a chosen small tolerance δ), at which point the midpoint is recognized as the root of the function.

The bisection method provides a robust numerical solution to finding the value of t that satisfies our equation, and it will give an approximate number of days until the portfolio reaches a value of α.

The algorithm is as follows:

0. Check if $f(0) < 0$. If true, then the wallet is already in liquidation. Return $t = 0$.
1. Check if $f(0) \times f(t_{\text{max}}) > 0$. If true, then there is no guaranteed root in the interval $[0, t_{\text{max}}]$. Return $t = \infty$.
2. Set $a = 0$ and $b = t_{\text{max}}$.
3. Set $c = \frac{a+b}{2}$.
4. If $|f(c)| < \epsilon$ or $|b - a| < \delta$ for some small tolerance values ϵ and δ, then c is your root. Terminate the algorithm. Return $t = c$.
5. If $f(a) \times f(c) < 0$, then the root is in the interval $[a, c]$. Set $b = c$ and go to step 4.
6. If $f(b) \times f(c) < 0$, then the root is in the interval $[c, b]$. Set $a = c$ and go to step 4.

The numeric solution provides a robust method to approximate the time until liquidation using a root finding algorithm such as the bisection method. While it may not offer the same precision as the analytic method, it circumvents some of the problems associated with the latter. It can handle a wider range of cases and provides a straightforward interpretation of the results, as it always returns real-valued solutions that coincide with the interpretation of time. This makes it an ideal choice for practical applications where ease of interpretation and robustness are key. Additionally, by appropriately setting the interval and the tolerance, the numeric method can provide a solution with a precision that is sufficient for many purposes.

4. Conclusion

The Anthias Liquip Score makes significant strides in advancing the methods of risk assessment in decentralized finance (DeFi) lending protocols. By challenging the limitations of the health factor metric, it offers a more nuanced understanding of liquidation risks faced by wallets in DeFi protocols.

We introduce the Anthias Liquip Score as a comprehensive and innovative measure of risk. This new metric draws from the variance and correlations of asset price actions in a lending position to provide a more accurate assessment of a wallet's risk exposure. Unlike the health factor, the Liquip Score is adept at gauging liquidation risk in scenarios involving varying levels of asset volatility and diverse combinations of collateral and debt.

Section 3.2's exploration of "Days Until Liquidation" underscores the practical implications of the Liquip Score. This measure, derived from the Liquip Score, provides tangible insights into the timeframe within which a wallet position might face liquidation, given its current circumstances and historical price action. Consequently, this complements the Liquip Score by enhancing the usability of risk assessments in the real-world context of DeFi protocols.

The Liquip Score extends beyond theoretical implications to provide practical utility for stakeholders in the DeFi ecosystem, including investors, borrowers, and lending platforms. By facilitating a more precise assessment of risk, it empowers these stakeholders to make informed decisions regarding their positions, thus fostering a more secure and robust financial landscape in the DeFi space.

The dynamic nature of the DeFi landscape calls for continuous evolution in risk assessment methodologies. Future research directions might explore more sophisticated distributional assumptions or incorporate additional variables influencing risk. While the Anthias Liquip Score presents a significant improvement in the current risk assessment methodologies, the journey towards a more complete understanding of risk in the DeFi space continues. These ongoing efforts will contribute to an increasingly sophisticated, accurate, and reliable set of risk assessment tools in the ever-evolving DeFi ecosystem.